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Extension of the Partition Sieve

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We demonstrate the correspondence which lies behind certain partition identities used by Andrews in his partition sieve. This leads to an extension of his methods and a generalization of his results.

1. INTRODUCTION

In 1971, Andrews [2; see also 1, 3] introduced his partition sieve in an attempt to obtain a combinatorial interpretation of Schur's [10] analytic proof of the Rogers–Ramanujan–Schur identities. (For a history of these identities, see Hardy [8, p. 91]). While his sieve yielded several new results, it was only partially successful in its original purpose. Two steps of his proof, the second involving the partition sieve, had to be proved analytically. In this paper, we give a combinatorial interpretation of his first step, and indicate how this leads to more general applications of the partition sieve.

The partition sieve relies on the notion of *successive rank*. Dyson [6] defined the *rank* of a partition to be the largest part minus the number of parts. Thus the rank of the partition $6+3+3+2+2+2+1$ of 19 is $6-7=-1$. If we represent this partition by its Ferrars graph:

$$\begin{array}{llllll} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & & & & & \end{array}$$

we see that the rank can also be defined as the number of nodes in the first row minus the number of nodes in the first column.

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DEFINITION. A *subgraph* of the Ferrars graph of a partition is that portion of the Ferrars graph which lies below a given row and to the right of a given column. The ith *proper subgraph* is the subgraph lying below the ith row and to the right of the ith column.

Atkin [4] has given the following definition:

DEFINITION. The ith *successive rank* (here denoted $\mathbf{SR}(i)$) of a partition is the rank of its $(i-1)$st proper subgraph.

For the preceding partition of 19, $\mathbf{SR}(1) = -1$, $\mathbf{SR}(2) = -3$, and $\mathbf{SR}(3) = 0$.

In his 1971 paper [2], Andrews announced the following result which arises out of the partition sieve. As he shows, for $M = 5$ and $r = 1$ or 2 it is equivalent to the Rogers–Ramanujan identities. The notation used here is different from that employed by Andrews.

THEOREM 1. *Given positive odd integer M and integral r, $0 < r < M/2$, let $A_{M,r}(n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm r \pmod{M}$. Let $B_{M,r}(n)$ denote the number of partitions of n whose successive ranks lie in the interval $[-r+2, M-r-2]$. Then $A_{M,r}(n) = B_{M,r}(n)$ for all n.*

In the proof of this theorem, Andrews uses the concept of *oscillations* of the successive ranks. In a notational change from Andrews, we make the following definitions:

DEFINITION. A partition has an *(a, b)-positive oscillation of length μ* if there exists a sequence $j_1 < j_2 < \cdots < j_\mu$ such that $\mathbf{SR}(j_1) \geqslant a-1$, $\mathbf{SR}(j_2) \leqslant -b+1$, $\mathbf{SR}(j_3) \geqslant a-1$, $\mathbf{SR}(j_4) \leqslant -b+1$, and so on.

DEFINITION. A partition has an *(a, b)-negative oscillation of length μ* if there exists a sequence $j_1 < j_2 < \cdots < j_\mu$ such that $SR(j_1) \leqslant -b+1$, $SR(j_2) \geqslant a-1$, $SR(j_3) \leqslant -b+1$, $SR(j_4) \geqslant a-1$, and so on.

DEFINITION. $p_{a,b}(\mu; n)$ denotes the number of partitions of n with an (a, b)-positive oscillation of length μ.

DEFINITION. $m_{a,b}(\mu; n)$ denotes the number of partitions of n with an (a, b)-negative oscillation of length μ.

The first step in proving Theorem 1 involves verifying the following four equations. $p(n)$ is the number of partitions of n.

If $a, b > 0$ have opposite parity and μ is odd, then

$$p_{a,b}(\mu; n) = p(n - \mu((\mu+1)a/2 + (\mu-1)b/2)), \tag{1.1}$$

$$m_{a,b}(\mu; n) = p(n - \mu((\mu+1)b/2 + (\mu-1)a/2)). \tag{1.2}$$

If $a, b > 0$ have opposite pairty and μ is even, then

$$p_{a,b}(\mu; n) = p(n - \tfrac{1}{2}\mu((\mu - 1)a + (\mu + 1)b)), \tag{1.3}$$

$$m_{a,b}(\mu; n) = p(n - \tfrac{1}{2}\mu((\mu - 1)b + (\mu + 1)a)). \tag{1.4}$$

Andrews proves these equations analytically, and asks if they have a combinatorial proof which shows the correspondence. We shall demonstrate this correspondence and exhibit several new results which arise out of it. We shall generalize Eqs. (1.1)–(1.4), first showing that the parity condition on a and b is unnecessary, and then extending these equations to cover more general types of oscillations. Finally, we shall prove

THEOREM 2. *Theorem 1 holds for any positive integer M.*

2. A LEMMA

We shall prove (1.1)–(1.4) without parity restrictions on a and b. We first observe that for each partition with an (a, b)-positive oscillation of length μ, its *conjugate* (the reflection over the main diagonal of its Ferrars graph) has a (b, a)-negative oscillation of length μ. Therefore, (1.1) and (1.2) are equivalent, as are (1.3) and (1.4).

Second, we claim that (1.1) and (1.3) are consequences of the following lemma:

LEMMA. *Let $r_{a,b}(\mu; n)$ denote the number of partitions $(d_1 + d_2 + \cdots + d_m)$ of n such that $d_i \geqslant d_{i+1}$ and for $1 \leqslant i \leqslant \mu$,*

$$\begin{aligned} d_{2i-1} - d_{2i} &\geqslant a, && \text{if } i \text{ is odd,} \\ &\geqslant b, && \text{if } i \text{ is even.} \end{aligned}$$

If $a, b > 0$, then $r_{a,b}(\mu; n) = p_{a,b}(\mu; n)$ for all n.

To see how (1.1) and (1.3) arise from this lemma, we consider a partition counted by $r_{a,b}(\mu; n)$. We subtract a from the largest part, then subtract b from each of the three largest parts, then subtract a from each of the five largest parts, then subtract b from each of the seven largest parts, and so on until we subtract a (if μ is odd) or b (if μ is even) from each of the $2\mu - 1$ largest parts. We are left with a partition of $n - \mu((\mu + 1)a + (\mu - 1)b)/2$ (if μ is odd) or of $n - \mu((\mu - 1)a + (\mu + 1)b)/2$ (if μ is even). Given integers μ, a, and b and a partition of $n - \mu((\mu + 1)a + (\mu - 1)b)/2$ (if μ is odd) or of $n - \mu((\mu - 1)a + (\mu + 1)b)/2$ (if μ is even), we reverse this process to give us a partition counted by $r_{a,b}(\mu; n)$. This establishes a one-to-one correspondence between partitions counted by $r_{a,b}(\mu; n)$ and the partition of $n - \mu((\mu + 1)a + (\mu - 1)b)/2$ (if μ is odd) or of $n - \mu((\mu - 1)a + (\mu + 1)b)/2$ (if μ is even). Thus, (1.1) and (1.3) follow from the lemma.

We shall prove this lemma by establishing a one-to-one correspondence between partitions counted by $r_{a,b}(\mu; n)$ and partitions counted by $p_{a,b}(\mu; n)$.

3. Proof of Lemma, Part I

In this section we demonstrate how to transform a partition counted by $p_{a,b}(\mu; n)$ into a partition counted by $r_{a,b}(\mu; n)$.

Step 1. Given positive integers μ, a, and b and a partition P with an (a, b)-positive oscillation of length μ, we define the following integers:

$$s = \max\{i \mid \mathbf{SR}(i) \geqslant a - 1 \text{ and the } i\text{th proper subgraph of } P \text{ has an } (a, b)\text{-negative oscillation of length } \mu - 1\}, \tag{3.1}$$

$$t = \max\{\mathbf{SR}(i) \mid 1 \leqslant i \leqslant s\}, \tag{3.2}$$

$$k = \min\{i \mid 1 \leqslant i \leqslant s, \mathbf{SR}(i) = t\}, \tag{3.3}$$

and, for $1 \leqslant j \leqslant s$,

$$c(j) = \min\{t - \mathbf{SR}(i) \mid 1 \leqslant i \leqslant j\}. \tag{3.4}$$

Remark. $t \geqslant SR(s) \geqslant a - 1$. For $j < k$, $c(j) > 0$. For $k \leqslant j \leqslant s$, $c(j) = 0$.

We now work with the Ferrars graph of P. For each j, $1 \leqslant j \leqslant s$, subtract $c(j)$ nodes from the jth column and add them to the $(j + 1)$st row (see Fig. 3.1). This yields a graph in which the number of nodes in the first row minus the number of nodes in the first column equals t, and only the second row may be longer than the row above it.

Furthermore, let $x(j)$ [$y(j)$] denote the number of nodes in the jth row [column] of our original graph. Then $\mathbf{SR}(j) = x(j) - y(j)$. By definition, $c(j) \leqslant c(j - 1)$. If $c(j) = c(j - 1)$, then $y(j) - c(j) \leqslant y(j - 1) - c(j - 1)$. If $c(j) < c(j - 1)$, then $c(j) = t - (x(j) - y(j))$, and therefore

$$\begin{aligned} y(j) - c(j) &= y(j) - (t - (x(j) - y(j))) \\ &= x(j) - t \\ &\leqslant x(j - 1) - t \\ &= y(j - 1) - (t - (x(j - 1) - y(j - 1))) \\ &\leqslant y(j - 1) - c(j - 1). \end{aligned}$$

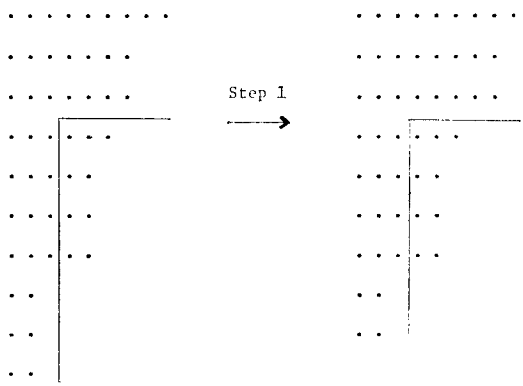

FIG. 3.1. Example for (1, 3)-positive oscillation of length 2; $s = 3$, $t = 0$, $k = 3$, $c(1) = 1$, $c(2) = 1$, $c(3) = 0$.

Thus, in the transformed graph, no column is longer than the column to the left of it.

Step 2. Consider the subgraph now lying below the first row. We conjugate it (see Fig. 3.2).

For the entire graph thus obtained, no column is longer than the column to the left of it, no row is longer than the row above it, and the difference between the number of nodes in the first and second rows equals $t + 1$ which is greater than or equal to a.

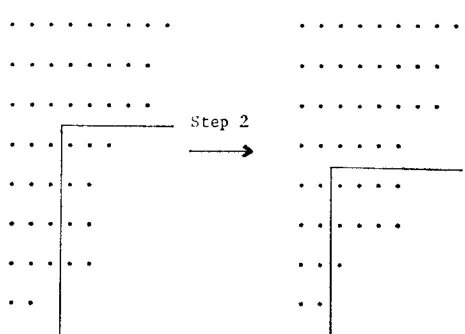

FIG. 3.2. Example for (1, 3)-positive oscillation of length 2.

We note that the subgraph which lies to the right of the $(s - 1)$st column and below the $(s + 1)$st row is the conjugate of the sth subgraph of the original Ferrars graph of P, and thus has a (b, a)-positive oscillation of length $\mu - 1$. Therefore, the subgraph lying below the second row has a (b, a)-positive oscillation of length $\mu - 1$.

We restrict out attention to this subgraph lying below the second row, and repeat steps 1 and 2 with the new parameters: $\mu - 1$, b, and a. This yields a new subgraph in which the difference between the number of nodes in its first and second rows in greater than or equal to b.

We continue this process through a total of 2μ steps. After every second step, we restrict our attention to the subgraph lying below the first two rows of the subgraph we have just transformed, and we apply steps 1 and 2 to this new subgraph.

At the completion of 2μ steps, we have a graph of a partition of n in which $d_i \geqslant d_{i+1}$ and for $1 \leqslant i \leqslant \mu$,

$$\begin{aligned} d_{2i-1} - d_{2i} &\geqslant a, && \text{if } i \text{ is odd,} \\ &\geqslant b, && \text{if } i \text{ is even.} \end{aligned}$$

4. Proof of Lemma, Part II

If we are given a partition of n which satisfies $d_i \geqslant d_{i+1}$ and for $1 \leqslant i \leqslant \mu$

$$\begin{aligned} d_{2i-1} - d_{2i} &\geqslant a, && \text{if } i \text{ is odd,} \\ &\geqslant b, && \text{if } i \text{ is even,} \end{aligned}$$

we want to be able to reverse the process described in Section 3 to give us a partition of n with an (a, b)-positive oscillation of length μ. This will establish a one-to-one correspondence between partitions counted by $r_{a,b}(\mu; n)$ and $p_{a,b}(\mu; n)$, and thus prove the lemma.

We begin with the Ferrars graph of the given partition and restrict our attention to the subgraph lying below the $(2\mu - 2)$th row.

Step 1. Given integers η, a, and b, and the graph of a partition for which the difference between the number of nodes in the first and second row is

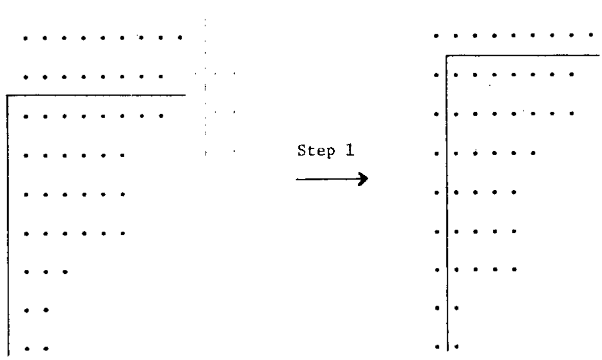

FIG. 4.1. Example for $a = 1$ and a subgraph with (3, 1)-positive oscillation of length 1.

greater than or equal to a, and the subgraph lying below the second row has a (b, a)-positive oscillation of length η, we conjugate the subgraph lying below the first row (see Fig. 4.1). (Note: initially $\eta = 0$.)

This obviously reverses step 2 of Section 3. The first proper subgraph of our graph now has an (a, b)-negative oscillation of length η, and the difference between the number of nodes in the first row and the first column is greater than or equal to $a - 1$. Also, no column is longer than the column to the left of it, and only the second row can be longer than the row above it.

Step 2. Let $x(i)$ [$y(i)$] denote the number of nodes in the ith row [column]. We define the following integers:

$$\begin{aligned} \sigma &= \max\{i \mid \mathbf{SR}(i) \geqslant a - 1, \text{ and the } i\text{th proper subgraph has an} \\ &\qquad (a, b)\text{-negative oscillation of length } \eta\}, & (4.1) \\ \tau &= x(1) - y(1), & (4.2) \\ \kappa &= \max\{i \mid 1 \leqslant i \leqslant \sigma, \mathbf{SR}(i) > \tau\} \qquad (=1 \text{ if this set is empty}), & (4.3) \end{aligned}$$

and, for $2 \leqslant j \leqslant \kappa$,

$$\gamma(j) = \max\{\mathbf{SR}(i) - \tau \mid j \leqslant i \leqslant \kappa\}. \tag{4.4}$$

Remark. $\mathbf{SR}(1) = \tau \geqslant a - 1$, and thus σ is well defined. $\gamma(j) \geqslant 0$.

For each j, $2 \leqslant j \leqslant \sigma$, subtract $\gamma(j)$ nodes from the jth row and add them to the $(j - 1)$st column (see Fig. 4.2). In the resulting graph, $\mathbf{SR}(\kappa) = \tau$ and $\mathbf{SR}(j) \leqslant \tau$ for $1 \leqslant j \leqslant \sigma$. No column is longer than the column to the left of it. The second row now has length

$$\begin{aligned} x(2) - \gamma(2) &\leqslant x(2) - (x(2) - y(2) - \tau) \\ &= y(2) + \tau \\ &= y(2) + x(1) - y(1) \\ &\leqslant x(1). \end{aligned}$$

We claim that no other row is longer than the row above it. By definition, $\gamma(j) \geqslant \gamma(j + 1)$. If $\gamma(j) = \gamma(j + 1)$, then $x(j) - \gamma(j) \geqslant x(j + 1) - \gamma(j + 1)$. If $\gamma(j) > \gamma(j + 1)$, then $\gamma(j) = \mathbf{SR}(j) - \tau = x(j) - y(j) - \tau$, and therefore

$$\begin{aligned} x(j) - \gamma(j) &= x(j) - (x(j) - y(j) - \tau) \\ &= y(j) + \tau \\ &\geqslant y(j + 1) + \tau \\ &= x(j + 1) - (x(j + 1) - y(j + 1) - \tau) \\ &\geqslant x(j + 1) - \gamma(j + 1). \end{aligned}$$

Therefore, step 2 leaves us with a graph of a partition with an (a, b)-positive oscillation of length $\eta + 1$.

To prove that step 2 reverses step 1 of Section 3, it is sufficient to prove that σ, τ, κ, and the $\gamma(j)$ equal s, t, k, and the $c(j-1)$, respectively.

From definitions (3.1) and (4.1), it is clear that $s = \sigma$. Since at the end of step 1 of Section 3, $SR(1) = t$, and at the end of step 2 of Section 4,

$$\max\{SR(i) \mid 1 \leqslant i \leqslant \sigma\} = \tau, \qquad \text{we see that } t = \tau.$$

Let $x(j)$ [$y(j)$] denote the number of nodes in the jth row [column] before applying step 1 of Section 3, and define $x'(j) = x(j) + c(j-1)$ [$y'(j) = y(j) - c(j)$], the number of nodes in the jth row [column] after applying step 1 of Section 3.

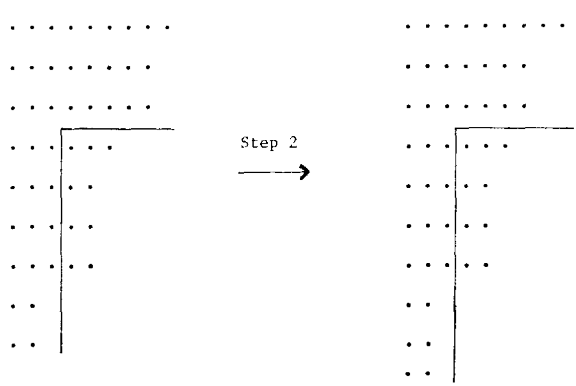

FIG. 4.2. Example for $a = 1$ and a subgraph with (3, 1)-positive oscillation of length 1; $\sigma = 3$, $\tau = 0$, $\kappa = 3$, $\gamma(2) = 1$, $\gamma(3) = 1$.

Now, $\kappa = \max\{i \mid x'(i) - y'(i) > \tau, 1 \leqslant i \leqslant \sigma\}$ (=1 if this set is empty). If $k \geqslant 2$, then

$$\begin{aligned} x'(k) - y'(k) &= x(k) + c(k-1) - (y(k) - c(k)) \\ &= x(k) - y(k) + c(k-1) \\ &> x(k) - y(k) = t = \tau. \end{aligned}$$

Since $\kappa \geqslant 1$, $\kappa \geqslant k$. Since $k \geqslant 1$, if $\kappa = 1$, then $\kappa = k$. If $\kappa > k \geqslant 1$, then $x'(\kappa) - y'(\kappa) > \tau$. But

$$\begin{aligned} x'(\kappa) - y'(\kappa) &= x(\kappa) + c(\kappa-1) - (y(\kappa) - c(\kappa)) \\ &= x(\kappa) - y(\kappa) \qquad (\text{since } \kappa > k) \\ &\leqslant t = \tau. \end{aligned}$$

Therefore, $\kappa = k$.

Now

$$\begin{aligned}\gamma(j) &= \max\{x'(i) - y'(i) - t \mid j \leqslant i \leqslant \kappa = k\} \\ &= \max\{x(i) + c(i-1) - y(i) + c(i) - t \mid j \leqslant i \leqslant k\} \\ &= \max\{c(i-1) + c(i) - (t - (x(i) - y(i))) \mid j \leqslant i \leqslant k\}.\end{aligned}$$

Since $c(i) \leqslant t - (x(i) - y(i))$,

$$\begin{aligned}\gamma(j) &\leqslant \max\{c(i-1) \mid j \leqslant i \leqslant k\} \\ &= c(j-1).\end{aligned}$$

Let h be the smallest integer such that $j \leqslant h \leqslant k$ and $c(h) = t - (x(h) - y(h))$. Such an h exists since $c(k) = t - (x(k) - y(k))$. Then

$$\begin{aligned}\gamma(j) &\geqslant x'(h) - y'(h) - t \\ &= c(h-1) + c(h) - (t - (x(h) - y(h))) \\ &= c(h-1) \\ &= c(h-2) \\ &\ \ \vdots \\ &= c(j-1).\end{aligned}$$

Thus, $\gamma(j) = c(j-1)$ for $2 \leqslant j \leqslant \sigma = s$. (Note: Since $s \geqslant k$, $c(s) = 0$.)

We have shown that step 2 of this section uniquely reverses step 1 of Section 3. If we continue to apply first step 1, then step 2 of this section, after every second step expanding our attention to include the two rows immediately above the subgraph we have just transformed, then after 2μ steps we obtain the graph of a partition of n with an (a, b)-positive oscillation of length μ. Furthermore, as we have shown, this uniquely reverses the process described in Section 3.

Thus, we have established a one-to-one correspondence between partitions counted by $r_{a,b}(\mu; n)$ and partitions counted by $p_{a,b}(\mu; n)$. This proves the lemma and so also proves (1.1)–(1.4) without parity restrictions on a and b.

5. Observations

The lemma in Section 2 can be extended. In the correspondence established in Sections 3 and 4, the largest part in a partition is left untouched. We therefore have

Corollary 1. *Let* $p_{a,b}(\mu; \lambda; n)$ $[r_{a,b}(\mu; \lambda; n)]$ *denote the number of* *partitions counted by* $p_{a,b}(\mu; n)$ $[r_{a,b}(\mu; n)]$ *which have largest part equal to* λ. *If* $a, b > 0$, *then*

$$r_{a,b}(\mu; \lambda; n) = p_{a,b}(\mu; \lambda; n) \qquad \textit{for all } n.$$

The proof given in Sections 3 and 4 will also work for a more generally defined oscillation.

DEFINITION. A partition has an $(a_1, a_2, \dots, a_\mu)$-*positive oscillation* if there exists a sequence $j_1 < j_2 < \cdots < j_\mu$ such that $S\mathbf{R}(j_1) \geqslant a_1 - 1$, $S\mathbf{R}(j_2) \leqslant -a_2 + 1$, $S\mathbf{R}(j_3) \geqslant a_3 - 1$, and, in general, if i is odd [even]

$$S\mathbf{R}(j_i) \geqslant a_i - 1 \qquad [\leqslant -a_i + 1].$$

DEFINITION. A partition has an $(a_1, a_2, \dots, a_\mu)$-*negative oscillation* if there exists a sequence $j_1 < j_2 < \cdots < j_\mu$ such that $S\mathbf{R}(j_1) \leqslant -a_1 + 1$, $S\mathbf{R}(j_2) \geqslant a_2 - 1$, $S\mathbf{R}(j_3) \leqslant -a_3 + 1$, and in general, if i is odd [even]

$$S\mathbf{R}(j_i) \leqslant -a_i + 1 \qquad [\geqslant a_i - 1].$$

DEFINITION. Let $p(a_1, a_2, \dots, a_\mu; n)$ denote the number of partitions of n with an $(a_1, a_2, \dots, a_\mu)$-positive oscillation.

DEFINITION. Let $m(a_1, a_2, \dots, a_\mu; n)$ denote the number of partitions of n with an $(a_1, a_2, \dots, a_\mu)$-negative oscillation.

Remark.

$$p_{a,b}(\mu; n) = p(a, b, a, b, \dots; n), \tag{5.1}$$

$$m_{a,b}(\mu; n) = m(b, a, b, a, \dots; n). \tag{5.2}$$

The proof of the lemma works equally well for an $(a_1, a_2, \dots, a_\mu)$-positive oscillation. We therefore have

COROLLARY 2. *Let* $r(a_1, a_2, \dots, a_\mu; n)$ *denote the number of partitions* $(d_1 + d_2 + \cdots + d_m)$ *of* n *such that* $d_i \geqslant d_{i+1}$ *and for* $1 \leqslant i \leqslant \mu$, $d_{2i-1} - d_{2i} \geqslant a_i$. *If* $a_i > 0$ *for all* a_i, *then* $r(a_1, a_2, \dots, a_\mu; n) = p(a_1, a_2, \dots, a_\mu; n)$ *for all* n.

From this corollary, we deduce the following equations

$$\begin{aligned} p(a_1, a_2, \dots, a_\mu; n) &= m(a_1, a_2, \dots, a_\mu; n) \\ &= p\left(n - \sum_{i=1}^{\mu} (2i - 1)\, a_i\right). \end{aligned} \tag{5.3}$$

Note that Eqs. (1.1)–(1.4) are special cases of (5.3).

6. Proof of Theorem 2

We follow Andrew's proof of Theorem 1 [1, 2, 3] using our notation. The only significant difference is that the parity restrictions on a and b have been removed.

Let $Q_{a,b}(n)$ denote the number of partitions of n such that all successive ranks lie in the interval $[-b+2, a-2]$. (Note: $Q_{a,b}(n) = B_{a+b,b}(n)$.) $Q_{a,b}(0) = 1$. Let

$$\mathcal{Q}_{a,b}(q) = \sum_{n=0}^{\infty} Q_{a,b}(n)\, q^n. \tag{6.1}$$

By Andrews' sieving technique [1, pp. 153–154]

$$Q_{a,b}(n) = p_{a,b}(0; n) + \sum_{\mu=1}^{\infty} (-1)^{\mu}\, p_{a,b}(\mu; n) + \sum_{\mu=1}^{\infty} (-1)^{\mu}\, m_{a,b}(\mu; n). \tag{6.2}$$

We now apply Eqs. (1.1)–(1.4),

$$\begin{aligned} Q_{a,b}(n) = p(n) &+ \sum_{\mu=1}^{\infty} p(n - \mu((2\mu-1)\, a + (2\mu+1)\, b)) \\ &- \sum_{\mu=1}^{\infty} p(n - (2\mu-1)(\mu a + (\mu-1)\, b)) \\ &+ \sum_{\mu=1}^{\infty} p(n - \mu((2\mu-1)\, b + (2\mu+1)\, a)) \\ &- \sum_{\mu=1}^{\infty} p(n - (2\mu-1)(\mu b + (\mu-1)\, a)). \end{aligned} \tag{6.3}$$

Define $(q)_\infty = \prod_{i=1}^{\infty} (1 - q^i)$. From (6.1) and (6.3) it follows that

$$\begin{aligned} \mathcal{Q}_{a,b}(q) &= (q)_\infty^{-1} \Bigg(1 + \sum_{\mu=1}^{\infty} q^{\mu((2\mu-1)a+(2\mu+1)b)} - \sum_{\mu=1}^{\infty} q^{(2\mu-1)(\mu a+(\mu-1)b)} \\ &\qquad + \sum_{\mu=1}^{\infty} q^{\mu((2\mu-1)b+(2\mu+1)a)} - \sum_{\mu=1}^{\infty} q^{(2\mu-1)(\mu b+(\mu-1)a)}\Bigg) \\ &= (q)_\infty^{-1} \left(\sum_{\mu=-\infty}^{\infty} q^{\mu((2\mu-1)a+(2\mu+1)b)} - \sum_{\mu=-\infty}^{\infty} q^{(2\mu-1)(\mu b+(\mu-1)a)} \right) \\ &= (q)_\infty^{-1} \left(\sum_{\mu\, \text{even}} q^{\mu((\mu-1)a+(\mu+1)b)/2} - \sum_{\mu\, \text{odd}} q^{\mu((\mu-1)a+(\mu+1)b)/2} \right) \end{aligned}$$

$$= (q)_\infty^{-1} \sum_{\mu=-\infty}^{\infty} (-1)^\mu q^{\mu((\mu-1)a+(\mu+1)b)/2}$$

$$= (q)_\infty^{-1} \sum_{\mu=-\infty}^{\infty} (-1)^\mu q^{\mu(a+b)/2+\mu(b-a)/2}$$

$$= (q)_\infty^{-1} \prod_{n=0}^{\infty} (1 - q^{(a+b)n+a})(1 - q^{(a+b)n+b})(1 - q^{(a+b)n+a+b}). \quad (6.4)$$

The last line uses Jacobi's triple product identity [9, 19.9.1].

If $a, b > 0$ and $a \neq b$, then the last expression is the generating function for $A_{a+b,b}(n)$, the number of partitions of n into parts $\not\equiv 0, \pm b \pmod{a+b}$. Thus

$$B_{a+b,b}(n) = Q_{a,b}(n) = A_{a+b,b}(n) \qquad \text{for all } n.$$

If we let $b = r$, $a = M - r$, then our restrictions on a and b become $0 < r < M$, $M \neq 2r$. For reasons of symmetry, we can restrict r to $0 < r < M/2$ without losing any generality. Therefore, if $0 < r < M/2$, then

$$A_{M,r}(n) = B_{M,r}(n)$$

for all n regardless of the parity of M. This is Theorem 2.

7. Conclusion

Let $C_{5,2}(n)$ denote the number of partitions of n into parts with minimal difference 2, and $C_{5,1}(n)$ denote the number of partitions of n into parts with minimal difference 2 and no part equal to 1. Andrews [1, 2, 3] has proved by exhibiting the correspondence that

$$B_{5,2}(n) = C_{5,2}(n) \qquad \text{for all } n, \quad (7.1)$$

$$B_{5,1}(n) = C_{5,1}(n) \qquad \text{for all } n. \quad (7.2)$$

If we compare this with Theorem 1, we see that

$$A_{5,2}(n) = C_{5,2}(n) \qquad \text{for all } n, \quad (7.3)$$

$$A_{5,1}(n) = C_{5,1}(n) \qquad \text{for all } n. \quad (7.4)$$

(7.3) and (7.4) are the Rogers–Ramanujan identities.

Let f_i denote the frequency of the part i in a given partition. For M odd, let $C_{M,r}(r)$ denote the number of partitions of n such that $f_1 \leqslant r-1$ and $f_i + f_{i+1} \leqslant (M-3)/2$. If $0 < r < M/2$ and M is odd, then Gordon's Theorem [7] states that

$$A_{M,r}(r) = C_{M,r}(n) \qquad \text{for all } n. \tag{7.5}$$

From this, we deduce that if $0 < r < M/2$ and M is odd, then

$$B_{M,r}(n) = C_{M,r}(n) \qquad \text{for all } n. \tag{7.6}$$

Andrews [1, 2, 3] has asked if there exists a correspondence which will independently prove (7.6).

Recently, Bressoud [5] announced the following result:

THEOREM 3. *Let $[a]$ denote the greatest integer $\leqslant a$. For arbitrary positive M, let $C_{M,r}(n)$ denote the number of partitions of n such that $f_1 \leqslant r-1$, $f_i + f_{i+1} \leqslant [(M-2)/2]$, and if $f_i + f_{i+1} = [(M-2)/2]$ then $if_i + (i+1)f_{i+1} \equiv r-1 \pmod{2 - M + 2[M/2]}$. For $0 < r < M/2$,*

$$A_{M,r}(n) = C_{M,r}(n) \qquad \textit{for all } n. \tag{7.7}$$

From Theorems 1 and 3, it follows that for $0 < r < M/2$,

$$B_{M,r}(n) = C_{M,r}(n) \qquad \text{for all } n. \tag{7.8}$$

Andrews' question can now be extended to include (7.8).

Finally, we ask whether the general oscillation defined in Section 5 can be used to equate other partition conditions with restrictions on successive ranks.

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